Spatial stability in linear thermoelasticity

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Abstract

Uniqueness and spatial stability are investigated for smooth solutions to boundary value problems in non-classical linearised and linear thermoelasticity subject to certain conditions on material coefficients. Uniqueness is derived for standard boundary conditions on bounded regions using a generalisation of Kirchhoff's method. Spatial stability is discussed for the semi-infinite prismatic cylinder in the absence of specified axial asymptotic behaviour. Alternative growth and decay estimates are established principally for the cross-sectional energy flux that is shown to satisfy a first order differential inequality. Uniqueness in the class of solutions with bounded energy follows as a corollary.

Separate discussion is required for the linearised and linear theories. Although the general approach is similar for both theories, the argument must be considerably modified for the treatment of the linear theory.

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Separate discussion is required for the linearised and linear theories. Although the general approach is similar for both theories, the argument must be considerably modified for the treatment of the linear theory.

1 Introduction

Spatial stability is a particular, but important, type of continuous data dependence. For boundary value problems on a bounded or unbounded region, the concept refers to the spatial behaviour of the solution with respect to distance from that part of the boundary on which non-zero data is specified. In most studies, behaviour cannot be measured pointwise, but instead must be analysed in some average sense, such as a cross-sectional L_2 norm. Uniqueness of the solution in the class of bounded energy usually is an easy consequence.

Various techniques for spatial stability, fully reviewed in [9, 7, 8], have been developed in the literature, originally for linear isothermal isotropic elasticity. Many examine behaviour on cylindrical regions and adopt a volume energy function as a measure of the solution. By contrast, the approach employed in the present study extends the procedure first developed in [4], and measures the solution by means of the cross-sectional energy flux. For simplicity, attention is confined to three-dimensional prismatic cylindrical regions, but our treatment explicitly includes nonhomogeneous anisotropic linearised and linear thermoelastostatic theories. The original investigation undertaken in [4] establishes alternative growth or decay behaviour, the respective rates indicating how far edge effects penetrate into the region. This feature is preserved in the present discussion for both the linearised and classical linear theories of thermoelastostatics.

In order for the paper to be reasonably self-contained, there is inevitable overlap with other publications. In particular, appropriate sections of the concise survey [15], which deals solely with the linearised theory, are reproduced and amplified not only for ease of reference, but also to facilitate comparison with the new proofs and results in the linear theory obtained in the paper.

Notation, introduced in Section 2, is either direct or indicial according to the particular context. A subscript comma denotes partial differentiation, while repeated Latin suffixes indicate summation over the range $1, 2, 3$. Repeated Greek subscripts denote summation over $1, 2$. Section 2 also states the boundary value problems to be considered in both the linearised and linear theories, explains their interrelation, and postulates certain sign-definite assumptions sufficient for the validity of subequent conclusions. Uniqueness of smooth solutions to standard boundary value problems in the linearised and linear theories is established for bounded regions in Section 3. Section 4 concerns non-classical linearised thermoelastostatics, and treats a non-homogeneous anisotropic material that occupies a semi-infinite cylinder, supposed prismatic for simplicity, in equilibrium under zero source terms (body force and heat supply), and subject to homogeneous lateral boundary data. The derivation and integration of a differential inequality to obtain respectively an exponentially increasing lower bound and an exponentially decreasing upper bound for alternative growth and decay of the cross-sectional energy flux expands the discussion presented in [15]. The decay estimate is employed to demonstrate uniqueness of the solution to the prescribed boundary value problem in the class of bounded total energy. Discussion of the linear theory, not presented previously in the literature and undertaken in Section 5, is generally similar to that for Section 4, but differs in important detail and requires significant modication of the proofs. The main difficulty occurs when the displacement gradient is necessarily replaced by the linear strain. Consequently, the quadratic form corresponding to the linear strain energy function is restricted to be positive-definite on the set of symmetric second order tensors. This in turn requires the introduction of a generalised Korn's inequality for the construction of the differential inequality governing the spatial evolution of the cross-sectional energy flux. The subsequent discussion proceeds as for the linearised theory, apart from the estimate of the amplitude in terms of the base data, which again involves important modification to the previous argument. Uniqueness in the class of bounded energy is implied by the decay estimate. A final section contains brief comment on spatial stability for non-cylindrical unbounded regions, and speculates on the general occurrence of algebraic decay rates. The interested reader may consult [15] for elaboration of these remarks.

Throughout, a solution is assumed to exist that is of sufficient smoothness to justify the calculations.

2 Notation and other preliminaries

Elements of linearised and linear thermoelastostatics relevant to the present study are presented in Sections 2.1 and 2.2, respectively. For either theory, the material in its reference configuration occupies a bounded or unbounded region Ω of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is supposed sufficiently smooth to admit application of the divergence theorem. The same rectangular coordinate system is used in the discussion of both theories.

2.1 Linearised thermoelastostatics

The deformation and thermal terms in the so-called *linearised* theory are derived as small perturbations of corresponding large quantities in the primary full nonlinear theory, and is otherwise known as the theory of *small deformations superposed upon large deformations.*

Let a point at X_i in the reference configuration Ω be deformed into points $x_i^{(1)}$ in the primary configuration and $x_i^{(2)}$ in the secondary configuration. The equilibrium equations for the increment between the large primary and secondary deformations whose respective components of the first Piola-Kirchhoff stress tensors are $t_{ij}^{(1)}, t_{ij}^{(2)}$ and of the heat fluxes are $q_i^{(1)}, q_i^{(2)}$, have the form

$$t_{ij,j} + \rho_0 f_i = 0, \qquad x \in \Omega, \tag{2.1}$$

$$q_{i,i} + \rho_0 s = 0, \qquad x \in \Omega, \tag{2.2}$$

where differentiation is with respect to $x_i^{(1)}$, ρ_0 is the mass density of the body occupying Ω, and $t_{ij} = t_{ij}^{(2)} - t_{ij}^{(1)}$, $q_i = q_i^{(2)} - q_i^{(1)}$. Furthermore, $f_i = f_i^{(2)} - f_i^{(1)}$ are components of the increment in the body force per unit mass, and $s = s^{(2)} - s^{(1)}$ is the increment in the scalar heat supply per unit mass. Superscripts $1, 2$ indicate that quantities belong to either the large primary or secondary states. The corresponding increments in the vector displacement $u_i = u_i^{(2)} - u_i^{(1)}$ and scalar temperature $\theta = \theta^{(2)} - \theta^{(1)}$ are supposed to be continuously differentiable. It is also supposed that t_{ij} and q_i can be expanded in series about the primary state and that terms of order higher than the first can be neglected. There is no loss in confusing spatial differentiation with respect to the primary configuration and that with respect to the secondary configuration to the order implied by linearisation. Accordingly, in what follows we set $x_i = x_i^{(1)} = x_i^{(2)}$ in linearised theories. These operations lead to coupled constitutive relations for the linearised increments of stress and heat flux given by (eg., [6])

$$t_{ij} = d_{ijkl} u_{k,l} + \beta_{ij}\theta, \qquad x \in \Omega, \tag{2.3}$$

$$q_i = h_{ijk} u_{j,k} + a_i\theta + k_{ij}\theta_{,j}, \qquad x \in \Omega, \tag{2.4}$$

where $\beta_{ij}(x)$ is the non-symmetric thermal coupling tensor, $k_{ij}(x)$ is the heat conduction tensor, $h_{ijk}(x)$ and $a_i(x)$ are other constitutive incremental tensors, and $d_{ijkl}(x)$ is the incremental elasticity tensor that possesses only the major symmetry

$$d_{ijkl} = d_{klij}.$$

The linear elasticities $c_{ijkl}(x)$ of the unstressed reference configuration possess both the major and minor symmetries

$$c_{ijkl} = c_{klij} = c_{jikl}, \tag{2.5}$$

and are related to the incremental elasticities by

$$d_{ijkl} = c_{ijkl} + \sigma_{ik}\delta_{jl},$$

where σ_{ij} is the symmetric Cauchy stress in the deformed primary configuration, and δ_{ij} is the usual Kronecker delta function.

Substitution of the constitutive relations (2.3) and (2.4) in the equilibrium equations (2.1) and (2.2) yields

$$\left(d_{ijkl}u_{k,l} + \beta_{ij}\theta\right)_{,j} + \rho_0 f_i \quad = \quad 0, \qquad x \in \Omega, \tag{2.6}$$

$$\left(h_{ijk}u_{j,k} + a_i\theta + k_{il}\theta_{,l}\right)_{,i} + \rho_0 s \quad = \quad 0, \qquad x \in \Omega, \tag{2.7}$$

to which must be adjoined prescribed kinematical and thermal boundary conditions.

Assume that Ω is bounded with smooth boundary $\partial\Omega$. For convenience, the boundary conditions mainly selected for study are those of *Dirichlet*:

$$u_i(x) = \bar{u}_i(x), \qquad \theta(x) = \bar{\theta}(x), \qquad x \in \partial\Omega, \tag{2.8}$$

where $\bar{u}_i(x)$ and $\bar{\theta}(x)$ are prescribed functions.

In general, the kinematical and thermal constitutive coefficients are differentiable functions of position x_i through dependence upon the primary configuration, or upon an inhomogeneous reference state. The same dependence implies that the coefficients d_{ijkl} become sign-definite only for certain primary configurations. Aspects of this topic for nonlinear elasticity are discussed in [16, Sections 53, 56].

When the temperature is uniform in the primary configuration ($\theta^{(1)}_{,i} = 0$), but the primary stress is non-zero, the incremental heat flux vector simplifies to

$$q_i = k_{ij}\theta_{,j}, \qquad x \in \Omega \tag{2.9}$$

and

$$h_{ijk} = 0, \qquad a_i = 0, \tag{2.10}$$

so that the heat conduction equation (2.7) reduces to

$$\left(k_{ij}\theta_{,i}\right)_{,j} + \rho_0 s = 0. \tag{2.11}$$

With special reference to the linearised theory, we suppose that:

(I) The thermal conductivity tensor is positive-definite in the sense that there exists a specified positive constant k_1 such that

$$k_1\zeta_i\zeta_i \leq k_{ij}\zeta_i\zeta_j. \tag{2.12}$$

It is immediately obvious that for the purposes of this assumption, there is no loss in supposing that k_{ij} is a symmetric tensor.

(II) There exists a given positive constant d_1 such that

$$d_1\xi_{ij}\xi_{ij} \leq d_{ijkl}\xi_{ij}\xi_{kl}, \qquad \forall\xi_{ij}. \tag{2.13}$$

We also impose two supplementary conditions, the first of which is

$$A \equiv k_1 - \left(\frac{\widetilde{a}^2}{\lambda_1(\Omega)}\right)^{1/2} > 0, \tag{2.14}$$

where

$$\widetilde{a}^2 = \max_{\Omega} a_i a_i, \tag{2.15}$$

and $\lambda_1(\Omega)$ is the first eigenvalue for the fixed membrane problem for Ω, so that

$$\lambda_1(\Omega)\int_{\Omega}\theta^2\,dx \leq \int_{\Omega}\theta_{,i}\theta_{,i}\,dx. \tag{2.16}$$

The second supplementary condition supposes that the coefficients β_{ij}, h_{ijk}, a_i, d_1, k_1 satisfy

$$F \equiv \lambda_1(\Omega)d_1^2A^2 - \widetilde{\beta}^2\,\widetilde{h}^2 > 0, \tag{2.17}$$

where

$$\widetilde{\beta}^2 = \max_{\Omega}\beta_{ij}\beta_{ij}, \tag{2.18}$$

$$\widetilde{h}^2 = \max_{\Omega} h_{ijk}h_{ijk}. \tag{2.19}$$

A necessary and sufficient condition for assumption (2.17) is

$$\lambda_1^{1/2}(\Omega)d_1A - \widetilde{\beta}\widetilde{h} > 0. \tag{2.20}$$

Assumptions (I) and (II) are consistent with physical experience. Inequality (2.12) states that the heat conductor tensor is positive-definite, which strengthens the non-negative condition derived from thermodynamics. Inequality (2.13), within the context of elastic stability theory, is sufficient for the dynamic stability of the null solution in linearised elastodynamics (cp., [16]). The supplementary conditions guarantee the sign-definiteness of a certain bilinear quadratic form vital in subsequent developments. Note that (2.17) holds trivially when the temperature in the primary configuration is uniform by virtue of (2.10).

Sign-definiteness in linear elastic theories cannot be deduced from thermodynamics with the exception, as just remarked, of the heat conduction tensor which is non-negative. Even arguments based upon stability, precisely or colloquially defined, are unreliable since often they are tautological. Moreover, in linearised theories, several coefficients depend crucially upon the primary state and therefore may become indefinite indicating, for example, the onset of bifurcation. Consequently, sign-definiteness must be introduced as a separate postulate.

2.2 Linear thermoelastostatics

The equilibrium equations for the linear theory may be derived from those for the linearised theory when the primary configuration has uniform temperature, zero stress, and null deformation so that $\theta^{(1)}$ is constant, and $u_i^{(1)} = \sigma_{ij} = 0$. The reference, primary and secondary states may be confused, and in the notation of Section 2.1 we may set $x_i = x_i^{(1)} = x_i^{(2)} = X_i$. Under these conditions, we have $d_{ijkl} = c_{ijkl}$, while the constitutive relations (2.3) and (2.4) become

$$
\begin{aligned}
t_{ij} &= c_{ijkl}e_{kl} + \beta_{ij}\theta, \qquad x \in \Omega, &(2.21)\\
q_i &= k_{ij}\theta_{,j}, \qquad x \in \Omega, &(2.22)
\end{aligned}
$$

where

$$
e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2.23}
$$

on noting the minor symmetry $(2.5)_2$. In the linear theory, the stress is symmetric, so that $t_{ij} = t_{ji}$, and the symmetries (2.5) together with (2.21) imply

$$
\beta_{ij} = \beta_{ji}. \tag{2.24}
$$

Substitution of the constitutive relations (2.21) in the equilibrium equations (2.1) leads to

$$
\left(c_{ijkl}e_{kl}\right)_{,j} + \left(\beta_{ij}\theta\right)_{,j} + \rho_0 f_i = 0, \qquad x \in \Omega. \tag{2.25}
$$

The thermal coefficients vanish in accordance with (2.10) and by (2.22), the heat conduction equation (2.2) reduces to the simplified form (2.11).

Boundary conditions necessary to complete the specification of the boundary value problem (2.25) and (2.11) are the same as those previously mentioned.

We retain the positive-definite assumption (2.12) for the heat conduction tensor, but replace condition (2.13) by the stricter requirement that the linear elasticities are positive-definite on the set of symmetric tensors, so that

$$
c_0\psi_{ij}\psi_{ij} \leq c_{ijkl}\psi_{ij}\psi_{kl}, \qquad \forall\psi_{ij} = \psi_{ji}, \tag{2.26}
$$

where c_0 is a given positive constant.

It will emerge that this condition demands significant modification of the proof of spatial stability, and to a lesser extent, of uniqueness. These topics are discussed in the next section.

3 Uniqueness on a bounded region

The systems of linear equations introduced in the previous section hold pointwise in Ω and consequently the solutions must be continuously differentiable to appropriate order. Otherwise the equations cannot be satisfied at all points of Ω. Such solutions are called *strong* or *classical* solutions in contrast to *weak* solutions which possess less differentiability and satisfy the equations only in a weak or averaged (i.e., integral) sense. Our concern here is solely with strong solutions which throughout are assumed to exist.

We seek conditions on the constitutive functions that ensure that the boundary value problems stated in Sections 2.1 and 2.2 possess a unique strong solution on bounded regions Ω. Uniqueness in unbounded regions is considered in subsequent sections. The conditions, which depend upon the type of boundary value problem and are imposed on the elastic moduli and elasticities respectively, usually are sufficient but not necessary, and generalise those employed in Kirchhoff's classical proof. We emphasise, however, that uniqueness may hold under conditions other than sign-definiteness.

As illustration of the technique, we discuss uniqueness of a strong solution to the Dirichlet boundary value problem. By linearity, uniqueness is equivalent to proving that at most the null solution exists subject to homogeneous boundary data and source terms. That is, for the purposes of the proof, we assume that $\tilde{u}_i = \tilde{\theta} = f_i = s = 0$.

3.1 Uniqueness in linearised thermoelastostatics

In this section, we prove the following theorem:

Theorem 3.1 . *Let us to assume that conditions (I) and (II) are satisfied and that the inequalities (2.14) and (2.17) hold. Then the boundary value problem (2.6) and (2.7) for presribed source terms and subject to (2.8) possesses a unique solution.*

Proof: For prescribed boundary data, body-force, and heat supply, consider two different solutions to the boundary value problem (2.6) and (2.7) and let u_i and θ denote the difference of the displacement and temperature between the two solutions respectively. Consequently, with this notation, we may set $f_i = s = 0$, and then multiply (2.6) by u_i, add to (2.7) multiplied by $\Gamma\theta$, where Γ is a positive constant to be chosen, and on integrate by parts over Ω, to obtain

$$\int_\Omega B(u_{i,j}, \theta_{,i}, \theta)\, dx = 0, \tag{3.1}$$

where

$$B(u_{i,j}, \theta_{,i}, \theta) = d_{ijkl}u_{i,j}u_{k,l} + \beta_{ij}u_{i,j}\theta + \Gamma h_{ijk}u_{j,k}\theta_{,i} + \Gamma a_i\theta_{,i}\theta + \Gamma k_{ij}\theta_{,i}\theta_{,j}. \tag{3.2}$$

An application of the Schwarz, Poincaré, and arithmetic-geometric mean inequalities yields the inequalities

$$\begin{aligned}
\pm \int_\Omega \beta_{ij}u_{i,j}\theta\, dx &\leq \frac{\widetilde{\beta}^2\alpha_1}{2}\int_\Omega u_{i,j}u_{i,j}\, dx + \frac{1}{2\alpha_1\lambda_1(\Omega)}\int_\Omega \theta_{,i}\theta_{,i}\, dx, && (3.3)\\
\pm \int_\Omega h_{ijk}u_{j,k}\theta_{,i}\, dx &\leq \frac{\alpha_2\widetilde{h}^2}{2}\int_\Omega u_{i,j}u_{i,j}\, dx + \frac{1}{2\alpha_2}\int_\Omega \theta_{,i}\theta_{,i}\, dx, && (3.4)\\
\pm \int_\Omega a_i\theta\theta_{,i}\, dx &\leq \frac{1}{2}\left(\widetilde{a}^2\alpha_3 + \frac{1}{\alpha_3\lambda_1(\Omega)}\right)\int_\Omega \theta_{,i}\theta_{,i}\, dx, && (3.5)
\end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary positive constants, and $\widetilde{a}, \widetilde{\beta}$ and $\widetilde{h}$ are defined by (2.15),(2.18), and (2.19) respectively.

After an appeal to assumptions (2.12) and (2.13), we have

$$\int_\Omega B\,dx \geq \int_\Omega \left(c_1 u_{i,j}u_{i,j} + c_2\theta_{,i}\theta_{,i}\right)\,dx, \tag{3.6}$$

in which the constants c_1, c_2 are given by

$$\begin{aligned} c_1 &= d_1 - \frac{\alpha_1\widetilde{\beta}^2}{2} - \frac{\Gamma\alpha_2\widetilde{h}^2}{2}, \\ c_2 &= \left(\Gamma k_1 - \frac{1}{2\alpha_1\lambda_1(\Omega)} - \frac{\Gamma}{2\alpha_2} - \frac{\Gamma}{2}\left(\widetilde{a}^2\alpha_3 + \frac{1}{\alpha_3\lambda_1(\Omega)}\right)\right). \end{aligned}$$

Let us set

$$\begin{aligned} \alpha_1\widetilde{\beta}^2 &= \alpha_2\Gamma\widetilde{h}^2 = \frac{(\widetilde{\beta}^2\widetilde{h}^2 + 3\lambda_1(\Omega)d_1^2A^2)}{4\lambda_1(\Omega)d_1A^2}, \\ \alpha_3 &= \left(\lambda_1(\Omega)\widetilde{a}^2\right)^{-1/2}, \\ \Gamma &= d_1A\widetilde{h}^{-2}, \end{aligned}$$

to recover

$$\begin{aligned} c_1 &= \frac{F}{4\lambda_1(\Omega)d_1A^2} > 0, \\ c_2 &= F\frac{d_1A^2}{\widetilde{h}^2(\widetilde{\beta}^2\widetilde{h}^2 + 3\lambda_1(\Omega)d_1^2A^2)} > 0, \end{aligned}$$

where $A(> 0)$ and $F(> 0)$ are defined respectively by (2.14) and (2.17). We finally conclude from (3.6) that

$$\int_\Omega B\,dx \geq 0, \tag{3.7}$$

with equality if and only if u_i, θ are identically zero in Ω. The condition (3.7) along with (3.1) implies that u_i and θ must vanish. Thus, only the trivial solution exists and uniqueness is established. $\square$

Note, in particular, that (2.12) and (2.13) are sufficient but not necessary conditions. Uniqueness also may hold under different sets of sufficient conditions; for example, when the quadratic form (3.2) is negative-definite, which may occur for certain primary stressed configurations.

As mentioned previously, when the temperature is uniform in the primary configuration, the governing equations reduce to the pair (2.6) and (2.11) and therefore become uncoupled. A strong solution to the heat conduction equation (2.11) exists and is unique for Dirichlet data provided the heat conduction tensor satisfies the positive-definite condition (2.12). The temperature therefore may be assumed known and consequently supplements the source terms in (2.6). A strong solution exists to the latter equation subject to Dirichlet data and the positive-definite condition (2.13) by the theory of elliptic equations. Moreover, homogeneous thermal data implies $\theta(x) \equiv 0$, and conditions on the coefficients d_{ijkl} for uniqueness depend solely upon those for the homogeneous equation

$$(d_{ijkl} u_{k,l})_{,j} = 0, \qquad x \in \Omega,$$

studied in [11].

Similar comments apply to the mixed and Neumann boundary value problems, although it must be observed that for the Neumann boundary value problem, the temperatue is unique only to within an arbitrary additive constant, while the displacement is unique only to within an arbitrary rigid body displacement.

3.2 Uniqueness in linear thermoelastostatics

The discussion of uniqueness for linear theories is similar to that for the linearised theory with one important exception. The coefficients d_{ijkl} are replaced by the fully symmetric elasticities c_{ijkl} that satisfy (2.5). Instead of using (2.6) in the operations leading to the energy equation (3.1), we combine (2.25) with (2.7) to obtain for zero source terms the expression

$$\int_\Omega \left(c_{ijkl} e_{ij} e_{kl} + \beta_{ij} u_{i,j} \theta + \Gamma h_{ijk} u_{j,k} \theta_{,i} + \Gamma a_i \theta_{,i} \theta + \Gamma k_{ij} \theta_{,i} \theta_{,j} \right) dx = 0,$$

where e_{ij} is defined by (2.23). While other assumptions and notation are unaltered, the positive-definite condition (2.13) is replaced by (2.26). The restriction of the quadratic form in (2.26) to symmetric tensors necessitates the introduction of *Korn's* inequality given by

$$\int_\Omega u_{i,j} u_{i,j} \, dx \leq C \int_\Omega e_{ij} e_{ij} \, dx,$$

where $C \geq 1$ is a positive constant dependent upon Ω. It is known (see, for example, [5, Sect.13]) that when u_i vanishes on the boundary $\partial\Omega$ then $C = 2$, but otherwise u_i must be normalised by the condition

$$\int_\Omega (u_{i,j} - u_{j,i}) \, dx = 0. \tag{3.8}$$

The uniqueness proof for the Dirichlet problem is easily adapted from that presented before, provided c_0 replaces d_1. For the Neumann problem, in order to use Korn's inequality, the solution must be normalised to exclude rigid body displacements (satisfied by (3.8)), and arbitrary additive constants in the temperature, but otherwise is unchanged.

4 Spatial stability in linearised thermoelastostatics

4.1 General introductory comments

Spatial behaviour of an equilibrium solution is analogous to time evolution of a dynamic solution. Indeed, Kirchhoff's exact analogy between a deformed rod in equilibrium and a spinning top (see [17]) may be generalised, for example, to a nonlinear elastostatic finite or semi-infinite three-dimensional prismatic cylinder in equilibrium subject to zero body forces and homogeneous lateral boundary conditions. In general, the equilibrium equations possess a quasi-Hamiltonian structure with respect to a preferred spatial variable that serves as a surrogate time variable. The alternative decay or growth of solutions described by the classic Phragmén-Lindelöf principle in elliptic (equilibrium) equations corresponds to Liapunov (asymptotic) stability and instability. In particular, decay relates to Saint-Venant's principle, and measures the persistence of effects due to boundary data and singularities. The reader may consult [10, Sect. 6.2] and [14, Sects. 14.6, 19.1.4] for further comment and references to the literature.

Different techniques, reviewed in [1] and [19], have been developed to investigate the more general problem of continuous data dependence, while particular methods for spatial behaviour are reviewed in [9, 7, 8]. Few if any of these treatments have been applied to thermoelastic problems, especially those considered here. The approach introduced in [4], and extended to classic linear thermoelastostatics in [18], is here selected to treat the semi-infinite prismatic cylinder $\Omega = D \times [0, \infty)$, where the cross-section $D \subset \mathbb{R}^2$ is a bounded domain which has boundary ∂D sufficiently smooth to admit application of the divergence theorem. We suppose that Ω is in equilibrium under zero source terms, and although the lateral surface may be subject to other standard types of boundary conditions, we study the homogeneous lateral boundary data specified by

$$u_i(x) = \theta(x) = 0, \qquad x \in \partial D \times [0, \infty), \tag{4.1}$$

together with the assumption that the displacement $u_i(x_\alpha, 0)$ and temperature $\theta(x_\alpha, 0)$ on the base $D(0)$ are pointwise prescribed to be

$$u_i(x_\alpha, 0) = w_i(x_\alpha), \qquad \theta(x_\alpha, 0) = \chi(x_\alpha), \quad x_\alpha \in D(0), \tag{4.2}$$

where w_i, χ are given functions that vanish on $\partial D(0)$.

The discussion in Section 4.2 elaborates and amends that presented in [15], and itself is a simplified version of that subsequently developed in Section 5 for the corresponding problem in linear thermoelastostatics. The latter treatment has not been published before.

4.2 Differential inequality

We construct an ordinary differential inequality for cross-sectional energy fluxes, rather than for volume measures of energy. In addition to conditions (2.12), (2.13), and simple modification of (2.14), we assume that the elasticity and heat conduction tensors are upper bounded on Ω, which permits the introduction of bounded positive constants $\widetilde{d}, \widetilde{k}$ defined by

$$\begin{aligned} \widetilde{d}^2 &= \max_{\Omega} d_{ijkl} d_{ijkl}, &(4.3)\\ \widetilde{k}^2 &= \max_{\Omega} k_{ij} k_{ij}. &(4.4)\end{aligned}$$

An appeal to Cauchy's inequality yields for vectors $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{R}^3$

$$\begin{aligned} d_{ijkl}\xi_i\xi_k\eta_j\eta_l &\leq \left(d_{ijkl}\xi_i\xi_k d_{pjql}\xi_p\xi_q\right)^{1/2}(\eta_r\eta_r)\\ &\leq \left(d_{ijkl}d_{ijkl}\left(\xi_p\xi_p\right)^2\right)^{1/2}(\eta_r\eta_r)\\ &\leq \widetilde{d}\xi_i\xi_i\eta_j\eta_j, \qquad x \in \Omega. &(4.5)\end{aligned}$$

Similarly, it may be established that

$$k_{ij}\xi_i\eta_j \leq \widetilde{k}\left(\xi_i\xi_i\right)^{1/2}\left(\eta_j\eta_j\right)^{1/2}. \tag{4.6}$$

Suppose that the Cartesian coordinate system is such that the x_3-axis is parallel to the cylinder's axis and the origin is located at a point in the cylinder's base. We consider the linear combination of energy fluxes over a cross-section $D(x_3)$, distance x_3 from the base, given by

$$H(x_3) = \int_{D(x_3)} t_{i3}u_i \, dS + \Lambda \int_{D(x_3)} q_3\theta \, dS, \tag{4.7}$$

where Λ is a positive constant to be determined.

An upper bound for $|H(x_3)|$ is obtained in part by employing inequalities corresponding to (3.3)-(3.5). Substitution from the constitutive relations (2.3) and (2.4), and subsequent appeal to the Schwarz and Poincaré inequalities, together with the upper bounds (4.5) and (4.6), successively yields

$$
\begin{aligned}
|H(x_3)| &\leq |\int_{D(x_3)} t_{ij} n_j u_i \, dS| + \Lambda |\int_{D(x_3)} q_i n_i \theta \, dS| \\
&= |\int_{D(x_3)} (d_{ijkl} u_{k,l} u_i n_j + \beta_{ij} u_i n_j \theta) \, dS| \\
&\quad + \Lambda |\int_{D(x_3)} \left(h_{ijk} u_{j,k} n_i \theta + a_i n_i \theta^2 + k_{ij} \theta_{,j} n_i \theta \right) \, dS| \\
&\leq \left[\int_{D(x_3)} d_{ijkl} u_{i,j} u_{k,l} \, dS \int_{D(x_3)} d_{ijkl} u_i n_j u_k n_l \, dS \right]^{1/2} \\
&\quad + \left[\int_{D(x_3)} u_i n_j u_i n_j \, dS \int_{D(x_3)} \beta_{ij} \beta_{ij} \theta^2 \, dS \right]^{1/2} \\
&\quad + \Lambda \left[\int_{D(x_3)} h_{ijk} u_{j,k} h_{ipq} u_{p,q} \, dS \int_{D(x_3)} n_i n_i \theta^2 \, dS \right]^{1/2} \\
&\quad + \Lambda \left[\int_{D(x_3)} (a_i a_i n_j n_j)^{1/2} \theta^2 \, dS \right] \\
&\quad + \Lambda \left[\int_{D(x_3)} k_{ij} \theta_{,i} \theta_{,j} \, dS \int_{D(x_3)} k_{ij} n_i n_j \theta^2 \, dS \right]^{1/2} \\
&\leq \left[\widetilde{d}^2 \int_{D(x_3)} u_{i,j} u_{i,j} \, dS \int_{D(x_3)} u_i u_i \, dS \right]^{1/2} + \left[\widetilde{\beta}^2 \int_{D(x_3)} u_i u_i \, dS \int_{D(x_3)} \theta^2 \, dS \right]^{1/2} \\
&\quad + \Lambda \left[\widetilde{h}^2 \int_{D(x_3)} u_{i,j} u_{i,j} \, dS \int_{D(x_3)} \theta^2 \, dS \right]^{1/2} + \Lambda \widetilde{a} \int_{D(x_3)} \theta^2 \, dS \\
&\quad + \Lambda \widetilde{k} \left[\int_{D(x_3)} \theta_{,i} \theta_{,i} \, dS \int_{D(x_3)} \theta^2 \, dS \right]^{1/2} \\
&\leq \left(\frac{\widetilde{d}^2}{\lambda_1} \right)^{1/2} \int_{D(x_3)} u_{i,j} u_{i,j} \, dS \\
&\quad + \left[\frac{\widetilde{\beta}}{\lambda_1} + \Lambda \left(\frac{\widetilde{h}^2}{\lambda_1} \right)^{1/2} \right] \left[\int_{D(x_3)} u_{i,j} u_{i,j} \, dS \int_{D(x_3)} \theta_{,i} \theta_{,i} \, dS \right]^{1/2} \\
&\quad + \Lambda \left[\frac{\widetilde{a}}{\lambda_1} + \frac{\widetilde{k}}{\lambda_1^{1/2}} \right] \int_{D(x_3)} \theta_{,i} \theta_{,i} \, dS,
\end{aligned}
$$

in which λ_1 denotes the first eigenvalue in the fixed membrane problem for the uniform cross-section D, and Λ is still to be chosen.

Application of the arithmetic-geometric mean inequality next gives

$$|H(x_3)| \le c_3 \int_{D(x_3)} u_{i,j}u_{i,j}\, dS + \Lambda c_4 \int_{D(x_3)} \theta_{,i}\theta_{,i}\, dS,$$

where the constants c_3, c_4 are given by

$$\begin{aligned} c_3 &= \left(\left(\frac{\widetilde{d}^2}{\lambda_1} \right)^{1/2} + \frac{\alpha_4}{2} \frac{\widetilde{\beta}}{\lambda_1} + \frac{\alpha_5 \Lambda}{2} \left(\frac{\widetilde{h}^2}{\lambda_1} \right)^{1/2} \right), \\ c_4 &= \left[\frac{1}{2\alpha_4 \Lambda} \frac{\widetilde{\beta}}{\lambda_1} + \frac{1}{2\alpha_5} \left(\frac{\widetilde{h}^2}{\lambda_1} \right)^{1/2} + \frac{\widetilde{a}}{\lambda_1} + \frac{\widetilde{k}}{\lambda_1^{1/2}} \right]. \end{aligned}$$

Among a variety of possible choices that lead to $c_3 = c_4$, we select the arbitrary positive constants α_4 and α_5 to satisfy

$$\begin{aligned} \frac{\widetilde{d}}{\lambda_1^{1/2}} + \frac{\alpha_4 \widetilde{\beta}}{2\lambda_1} &= \frac{\widetilde{\beta}}{2\alpha_4 \Lambda \lambda_1}, \\ \frac{\alpha_5 \Lambda \widetilde{h}}{2\lambda_1^{1/2}} &= \frac{\widetilde{h}}{2\alpha_5 \lambda_1^{1/2}} + \frac{b}{\lambda_1^{1/2}}, \end{aligned}$$

where

$$b = \left(\frac{\widetilde{a}}{\lambda_1^{1/2}} + \widetilde{k} \right). \tag{4.8}$$

Appropriate solutions to these equations are

$$\begin{aligned} \alpha_4 &= \frac{1}{\widetilde{\beta}} \left[\left(\widetilde{d}^2 + \widetilde{\beta}^2 \Lambda^{-1} \right)^{1/2} - \widetilde{d} \right], \\ \alpha_5 &= \frac{1}{\Lambda \widetilde{h}} \left[\sqrt{\left(b^2 + \Lambda \widetilde{h}^2 \right)} + b \right], \end{aligned}$$

which lead to the sought upper bound

$$|H(x_3)| \le c_3 \left(\int_{D(x_3)} u_{i,j}u_{i,j}\, dS + \Lambda \int_{D(x_3)} \theta_{,i}\theta_{,i}\, dS \right), \tag{4.9}$$

where the positive constant c_3 is given by

$$c_3 = \frac{1}{2\lambda_1^{1/2}} \left[\widetilde{d} + \left(\widetilde{d}^2 + \widetilde{\beta}^2 \Lambda^{-1} \lambda_1^{-1} \right)^{1/2} + b + \left(b^2 + \Lambda \widetilde{h}^2 \right)^{1/2} \right].$$

Next, integration by parts and use of (2.6), (2.7) with zero source terms, and the homogeneous lateral boundary conditions (4.1) leads to the expressions

$$\begin{aligned} H(x_3) - H(y_3) &= \int_{y_3}^{x_3} I(\eta)\, d\eta, \qquad 0 \le y_3 < x_3 \le \infty, \\ H'(x_3) &= I(x_3), \end{aligned} \tag{4.10}$$

where a superposed prime indicates differentiation with respect to x_3, and

$$I(x_3) = \int_{D(x_3)} B(u_{i,j}, \theta_{,i}, \theta)\, dS.$$

The quadratic form B is given by (3.2) but with Γ replaced by Λ. Subject to (2.12)-(2.17) with λ_1 written for $\lambda_1(\Omega)$, we may perform operations similar to those leading to (3.6) to obtain

$$I(x_3) \geq \int_{D(x_3)} \left(c_5 u_{i,j} u_{i,j} + \Lambda c_6 \theta_{,i}\theta_{,i}\right)\, dS, \tag{4.11}$$

where the constants c_5 and c_6 are defined by

$$\begin{aligned} c_5 &= \left(d_1 - \frac{\alpha_6 \widetilde{\beta}^2}{2\lambda_1} - \frac{\alpha_7 \Lambda \widetilde{h}^2}{2}\right), \\ c_6 &= \left(A - \frac{1}{2\Lambda\alpha_6} - \frac{1}{2\alpha_7}\right), \end{aligned}$$

where α_6, α_7 are arbitrary positive constants, and $A > 0$ is given by (2.14), but with $\lambda_1(\Omega)$ replaced by λ_1.

Set

$$\frac{\alpha_6 \widetilde{\beta}^2}{\lambda_1} = \alpha_7 \Lambda \widetilde{h}^2 = \alpha_8,$$

and select Λ, α_8 to ensure that the coefficients c_5, c_6 are positive and satisfy $c_5 = c_6$. In view of assumptions (2.12)-(2.14), we choose

$$\begin{aligned} \Lambda &= \frac{\widetilde{\beta}}{\lambda_1^{1/2} \widetilde{h}}, \\ Q &= d_1 - A, \end{aligned} \tag{4.12}$$

so that α_8 satisfies the quadratic equation

$$\alpha_8^2 - Q\alpha_8 - \widetilde{\beta}\widetilde{h}\lambda_1^{-1/2} = 0,$$

whose positive root is given by

$$\alpha_8 = \frac{1}{2}\left[Q + \sqrt{\left(Q^2 + 4\widetilde{\beta}\widetilde{h}\lambda_1^{-1/2}\right)}\right]. \tag{4.13}$$

The condition corresponding to (2.20) implies

$$(A + d_1)^2 - (d_1 - A)^2 = 4 d_1 A > 4\widetilde{\beta}\widetilde{h}\lambda^{-1/2},$$

and consequently

$$\begin{aligned} \sqrt{\left(Q^2 + 4\widetilde{\beta}\widetilde{h}\lambda^{-1/2}\right)} &< A + d_1 \\ &= (A - d_1) + 2d_1, \end{aligned}$$

which from (4.13) implies $\alpha_8 < d_1$. Subject to these conditions, we have

$$\begin{aligned} c_5 &= d_1 - \alpha_8 \\ &= \frac{1}{2}\left[(d_1 + A) - \sqrt{\left(Q^2 + 4\widetilde{\beta}\widetilde{h}\lambda_1^{-1/2}\right)}\right]. \end{aligned}$$

We conclude that

$$H'(x_3) = I(x_3) \geq c_5 \int_{D(x_3)} (u_{i,j}u_{i,j} + \Lambda\theta_{,i}\theta_{,i})\, dS \geq 0, \qquad x_3 \geq 0. \tag{4.14}$$

On combining (4.9) and (4.14), we obtain the required fundamental differential inequality in the form

$$|H(x_3)| \leq \alpha^{-1} H'(x_3), \qquad 0 \leq x_3 \leq \infty, \tag{4.15}$$

where the constant $\alpha = c_5/c_3$ is positive.

4.3 Growth and decay estimates

To extract information from (4.15), we employ the method developed in [4], and first suppose there exists $y_3 \geq 0$ such that $H(y_3) > 0$, which by (4.14) implies that $H(x_3) > 0$ for $x_3 \geq y_3 \geq 0$. The appropriate component of the fundamental inequality (4.15) becomes

$$\alpha H(x_3) - H'(x_3) \leq 0, \qquad y_3 \leq x_3 \leq \infty,$$

which on integration yields

$$H(x_3) \geq H(y_3) \exp \alpha(x_3 - y_3), \qquad y_3 \leq x_3 \leq \infty. \tag{4.16}$$

We conclude that $H(x_3) \to \infty$ as $x_3 \to \infty$. We prove that this result leads in turn to the asymptotic unboundedness (as $x_3 \to \infty$) of the energy function $E(y_3, x_3)$ defined to be

$$E(y_3, x_3) = \int_{\Omega(y_3,x_3)} (u_{i,j}u_{i,j} + \Lambda\theta_{,i}\theta_{,i})\, dx, \qquad 0 \leq y_3 \leq x_3 \leq \infty, \tag{4.17}$$

where

$$\Omega(y_3, x_3) = \{z \in \Omega : y_3 \leq z_3 \leq x_3\}, \tag{4.18}$$

and Λ is specified by (4.12).

It follows from (4.9) that for $0 \leq y_3 \leq x_3 \leq \infty$, we have

$$\begin{aligned} c_3 E(y_3, x_3) &\geq \int_{y_3}^{x_3} H(\eta)\, d\eta \\ &\geq \frac{H(y_3)}{\alpha} \left(\exp \alpha(x_3 - y_3) - 1\right), \end{aligned}$$

which proves the assertion on letting $x_3 \to \infty$.

On the other hand, let us suppose that the function $E(x_3, \infty)$ is bounded for all $x_3 \geq 0$, so that $H(x_3) \leq 0$, $0 \leq x_3 \leq \infty$. Then, from (4.15) we obtain

$$H'(x_3) + \alpha H(x_3) \geq 0, \qquad 0 \leq x_3 \leq \infty,$$

which after integration gives

$$-H(x_3) \leq -H(0) \exp(-\alpha x_3), \qquad 0 \leq x_3 \leq \infty. \tag{4.19}$$

We conclude that $H(x_3) \to 0$ as $x_3 \to \infty$. We may alternatively express (4.19) in terms of the energy function $E(y_3, x_3)$, defined in (4.17), on first noting from (4.10) that

$$-H(x_3) = \int_{x_3}^{\infty} I(\eta)\, d\eta,$$

so that by (4.14) and (4.17) we have

$$-H(x_3) \geq c_5 E(x_3, \infty).$$

Again, integration by parts applied to (4.7) yields

$$\begin{aligned} -H(0) &= \int_{D(0)} t_{ij} u_i n_j \, dS + \Lambda \int_{D(0)} q_i n_i \theta \, dS \\ &= \int_\Omega t_{ij} u_{i,j} \, dx + \Lambda \int_\Omega q_i \theta_{,i} \, dx, \end{aligned} \tag{4.20}$$

where Λ is still given by (4.12). Calculations similar to those used to deduce (4.9), show that after substitution from (2.3) and (2.4) we obtain from (4.20) the bound

$$\begin{aligned} -H(0) &= \int_\Omega \left(d_{ijkl} u_{i,j} u_{k,l} + \beta_{ij} u_{i,j} \theta\right) \, dx \\ &\quad + \Lambda \int_\Omega \left(h_{ijk} u_{j,k} \theta_{,i} + a_i \theta \theta_{,i} + k_{ij} \theta_{,i} \theta_{,j}\right) \, dx \\ &\leq \widetilde{d} \int_\Omega u_{i,j} u_{i,j} \, dx + \widetilde{\beta} \left[\int_\Omega u_{i,j} u_{i,j} \, dx \int_\Omega \theta^2 \, dx\right]^{1/2} \\ &\quad + \Lambda \widetilde{h} \left[\int_\Omega u_{i,j} u_{i,j} \, dx \int_\Omega \theta_{,i} \theta_{,i} \, dx\right]^{1/2} + \Lambda \widetilde{a} \left[\int_\Omega \theta^2 \, dx \int_\Omega \theta_{,i} \theta_{,i} \, dx\right]^{1/2} \\ &\quad + \Lambda \widetilde{k} \int_\Omega \theta_{,i} \theta_{,i} \, dx. \end{aligned}$$

Poincaré's inequality yields

$$\begin{aligned}
\int_\Omega \theta^2\,dx &= \int_0^\infty \int_{D(\eta)} \theta^2\,dSd\eta \\
&\leq \int_0^\infty \left(\lambda_1^{-1}\int_{D(\eta)} \theta_{,\alpha}\theta_{,\alpha}\,dS\right)\,d\eta \\
&\leq \int_0^\infty \left(\lambda_1^{-1}\int_{D(\eta)} \theta_{,i}\theta_{,i}\,dS\right)\,d\eta \\
&= \lambda_1^{-1}\int_\Omega \theta_{,i}\theta_{,i}\,dx,
\end{aligned}$$

which on application of the arithmetic-geometric mean inequality leads to

$$-H(0) = c_7\int_\Omega u_{i,j}u_{i,j}\,dx + \Lambda c_8\int_\Omega \theta_{,i}\theta_{,i}\,dx,$$

where Λ is specified by (4.12), and for arbitrary positive constant α_9,

$$\begin{aligned}
c_7 &= \left[\widetilde{d} + \frac{L\alpha_9}{2}\right], \\
c_8 &= \left[\frac{1}{2\alpha_9}\left(\frac{(1+\Lambda)}{\Lambda}\right) + b\right].
\end{aligned}$$

The constant b is given by (4.8) while the constant L is given by

$$\begin{aligned}
L &= \frac{\widetilde{\beta}^2}{\lambda_1} + \Lambda\widetilde{h}^2 \\
&= (1+\Lambda)\,\frac{\widetilde{\beta}\widetilde{h}}{\lambda_1^{1/2}}.
\end{aligned}$$

We choose α_9 such that $c_7 = c_8$, and consequently have

$$\alpha_9^2 L + 2\alpha_9\left(\widetilde{d} - b\right) - \left(1 + \Lambda^{-1}\right) = 0,$$

whose positive root is

$$\alpha_9 = \frac{\left(b - \widetilde{d}\right) + \sqrt{\left(b - \widetilde{d}\right)^2 + \widetilde{h}^2\,(1+\Lambda)^2}}{L}.$$

We conclude that

$$c_7 = \frac{1}{2}\left[\widetilde{d} + b + \sqrt{\left(b - \widetilde{d}\right)^2 + \widetilde{h}^2\,(1+\Lambda)^2}\right]$$

and finally that

$$-H(0) \leq c_7 E(0,\infty).$$

Consequently, we may rewrite (4.19) as

$$c_5 E(x_3, \infty) \leq -H(x_3) \leq c_7 E(0, \infty) \exp(-\alpha x_3). \tag{4.21}$$

We have proved:

Theorem 4.1 *Let us assume that conditions (I) and (II) are satisfied and that inequalities (2.14), (2.17), (4.5) and (4.6) hold. Then on the prismatic semi-infinite cylinder, the solution to the linear system (2.6) and (2.7) subject to zero source terms and homogeneous lateral boundary conditions (4.1) satisfies either the growth estimate (4.16) or the decay estimates (4.19) and (4.21).*

Uniqueness is an immediate corollary of this Theorem.

Corollary 4.1 (Uniqueness) *In the class of bounded energies $E(0, \infty)$ there is at most one solution to the boundary value problem (2.6), (2.7) on the semi-infinite prismatic cylinder subject to conditions stipulated in Theorem 4.1.*

Proof: First, note that by hypothesis, $H(x_3) \leq 0$ for $x_3 \geq 0$. It must be proved that at most only the trivial solution exists for homogeneous data. Thus, we assume that $H(0) = 0$ and from (4.19) deduce that $H(x_3) = 0$ for $0 \leq x_3 \leq \infty$, which by (4.21) implies $E(x_3, \infty) = 0, x_3 \geq 0$. Consequently, $u_{i,j} = \theta_{,i} = 0$ for $x \in D \times [0, \infty)$ and so by continuity and the boundary conditions, we conclude that $u_i = \theta = 0$ for $x \in D \times [0, \infty)$. The proof is complete. □

4.4 The amplitude term

A full description of the decay estimate requires an upper bound for the amplitudes, $-H(0)$ or $E(0)$, in terms of base data.

For this purpose, we introduce smooth functions $v_i(x), \phi(x)$ that satisfy the boundary conditions

$$\begin{aligned} v_i(x) &= \phi(x) = 0, \qquad x \in \partial D \times [0, \infty), & (4.22)\\ v_i(x) &= w_i(x_\alpha), \quad \phi(x) = \chi(x_\alpha), \qquad x \in D(0), & (4.23) \end{aligned}$$

where the Dirichlet base data functions are specified in (4.2). We also suppose the asymptotic behaviour

$$v_i(x) \to 0, \quad \phi(x) \to 0, \qquad x_3 \to \infty, \tag{4.24}$$

and that all source terms vanish.

On multiplying (2.6) by v_i, adding to (2.7) multiplied by $\Lambda\phi$ for positive constant Λ defined by (4.12), integrating by parts over Ω, and recalling (4.11), we obtain after some rearrangement

$$\begin{aligned} c_5 E(0, \infty) &\leq \int_\Omega B\, dx \\ &= \int_\Omega (d_{ijkl} v_{i,j} u_{k,l} + \beta_{ij} \theta v_{i,j} + \Lambda h_{ijk} u_{j,k} \phi_{,i} + \Lambda a_i \theta \phi_{,i} + \Lambda k_{ij} \theta_{,j} \phi_{,i})\, dx. \end{aligned}$$

Standard use of the Schwarz, Poincaré, and arithmetic-geometric mean inequalities next yields

$$c_5 E(0,\infty) \le c_9 \int_\Omega v_{i,j} v_{i,j}\, dx + \Lambda c_{10} \int_\Omega \phi_{,i}\phi_{,i}\, dx, \tag{4.25}$$

where

$$\begin{aligned} c_9 &= \left(\frac{\widetilde{d}^2 \alpha_{10}}{2} + \frac{\widetilde{\beta}^2 \alpha_{11}}{2\lambda_1} \right), \\ c_{10} &= \left(\frac{\Lambda \widetilde{h}^2 \alpha_{12}}{2} + \frac{\widetilde{a}^2 \alpha_{13}}{2\lambda_1} + \frac{\widetilde{k}^2 \alpha_{14}}{2} \right), \end{aligned}$$

in which $\widetilde{d}$ and $\widetilde{k}$ are defined in (4.3) and (4.4) respectively, and the arbitrary positive constants $\alpha_i, i = 10, \ldots 14$ are chosen to be

$$\begin{aligned} \alpha_{10} &= \alpha_{12} = 2c_5^{-1}, \\ \Lambda\alpha_{11} &= \alpha_{13} = \alpha_{14} = 3c_5^{-1}. \end{aligned}$$

Hence, we have

$$\begin{aligned} c_9 &= \frac{1}{2\lambda_1 \Lambda c_5} \left(2\lambda_1 \Lambda \widetilde{d}^2 + 3\widetilde{\beta}^2 \right), \\ c_{10} &= \frac{1}{2\lambda_1 c_5} \left(2\lambda_1 \Lambda \widetilde{h}^2 + 3\widetilde{a}^2 + 3\lambda_1 \widetilde{k}^2 \right) \end{aligned}$$

We now select the functions v_i, ϕ to be

$$v_i(x) = u_i(x_\alpha, 0) \exp(-s x_3) = w_i(x_\alpha) \exp(-s x_3), \tag{4.26}$$

$$\phi(x) = \theta(x_\alpha, 0) \exp(-s x_3) = \chi(x_\alpha) \exp(-s x_3), \tag{4.27}$$

where s is a positive constant to be determined, and the functions w_i, χ are prescribed. Substitution in (4.25) and optimisation with respect to s leads to the required estimate

$$c_5 E(0,\infty) \le (D_1 D_2)^{1/2}, \tag{4.28}$$

where the data terms D_1, D_2 are given by

$$\begin{aligned} D_1 &= c_9 \int_D w_{i,\beta} w_{i,\beta}\, dS + \Lambda c_{10} \int_D \chi_{,\beta}\chi_{,\beta}\, dS, \\ D_2 &= c_9 \int_D w_i w_i\, dS + \Lambda c_{10} \int_D \chi^2\, dS. \end{aligned}$$

4.5 Additional decay behaviour

We may deduce from inequality (4.21) that the mean-square over $D(x_3)$ of the displacement and temperature, and of the displacement and temperature gradient, exhibit similar exponential decay. Note also that the derivation of the basic inequality (4.15) is valid irrespective of the cylinder's length, and in particular holds for an infinite cylinder. We infer that on the infinite cylinder only the trivial solution can exist in the class of bounded stored energies.

The previous analysis includes the special case of a uniform primary temperature when $\widetilde{h} = \widetilde{a} = 0$. In consequence, only inequalities (2.12), (2.13), (4.5), and (4.6) are needed to establish conclusions similar to those just derived.

5 Spatial stability for linear thermoelastostatics

5.1 Introduction

We now investigate spatial stability for the linear theory of thermoelastostatics represented by the system (2.25) and (2.11) subject to appropriate boundary conditions. For simplicity, we continue to consider the semi-infinite prismatic cylinder $\Omega = D \times [0, \infty)$, whose lateral boundary $\partial D \times [0, \infty)$ is sufficiently smooth to admit application of the divergence theorem. A Cartesian system of rectangular coordinates is again chosen with origin in the base of the cylinder and positive x_3-axis directed along that of the cylinder. In addition, $D(x_3)$ denotes the cross-section at distance x_3 from the cylinder's base, and we define partial volumes $\Omega(y_3, x_3)$ of Ω by (4.18). In this Section, however, we suppose that Ω is occcupied by a linear thermoelastic material in equilibrium under zero body force and heat supply, and loaded only on the base $D(0)$ with the remaining lateral surface subject to the homogeneous data

$$\begin{aligned} t_{i\alpha} n_\alpha u_i &= 0, \qquad x \in \partial D \times [0, \infty), && (5.1) \\ k_{\alpha\beta} \theta_{,\alpha} n_\beta \theta &= 0, \qquad x \in \partial D \times [0, \infty), && (5.2) \end{aligned}$$

where n_i are the Cartesian components of the unit outward normal on $\partial\Omega$.

The linear elasticities c_{ijkl} satisfy the major and minor symmetries (2.5) and are positive-definite in the sense of inequality (2.26). Likewise, we suppose that the heat conduction tensor k_{ij} satisfies the positive-definite condition (2.12).

Proofs are completed in detail only for homogeneous Dirichlet data on the lateral surface, while those for Neumann data are only sketched. It should be emphasised that asymptotic behaviour is not prescribed for large axial distance, but, as in Section 4, emerges as a consequence of the arguments. As before, we establish and integrate a differential inequality for the cross-sectional energy flux $H(x_3)$, defined in (4.7), but with the constant Λ not necessarily given by (4.12). Significant modification of the method described in Section 4.2 is required due to the positive condition (2.26) being restrictied to symmetric tensors.

The special example of a nonhomogeneous isotropic linear thermoelastic rectangular strip considered in [2] uses the Airy stress function and second order differential inequalities to establish decay rates by a different method to that developed here.

5.2 Differential inequality

Consider the previously introduced cross-sectional flux (4.7), repeated here for convenience,

$$H(x_3) = \int_{D(x_3)} t_{i3} u_i \, dS + \Lambda \int_{D(x_3)} q_3 \theta \, dS, \tag{5.3}$$

in which the positive constant Λ is not specified by (4.12). Integration by parts over $\Omega(y_3, x_3)$, appeal to the equilibrium equations (2.25) and (2.11), subject to vanishing body force and heat supply, and the lateral boundary conditions (5.1) and (5.2), successively yields

$$\begin{aligned}
H(x_3) - H(y_3) &= \int_{\Omega(y_3,x_3)} (t_{i3} u_i)_{,3} \, dx + \Lambda \int_{\Omega(y_3,x_3)} (q_3 \theta)_{,3} \, dx \\
&= \int_{\Omega(y_3,x_3)} \left(-t_{i\alpha,\alpha} u_i + t_{i3} u_{i,3}\right) \, dx \\
&\quad + \Lambda \int_{\Omega(y_3,x_3)} \left(-q_{\alpha,\alpha} \theta + q_3 \theta_{,3}\right) \, dx \\
&= \int_{\Omega(y_3,x_3)} t_{ij} u_{i,j} \, dx + \Lambda \int_{\Omega(y_3,x_3)} q_i \theta_{,i} \, dx + H(y_3) \\
&= \int_{\Omega(y_3,x_3)} \left(c_{ijkl} e_{kl} u_{i,j} + \beta_{ij} u_{i,j} \theta\right) \, dx + \Lambda \int_{\Omega(y_3,x_3)} k_{ij} \theta_{,i} \theta_{,j} \, dx \\
&= \int_{\Omega(y_3,x_3)} \left(c_{ijkl} e_{ij} e_{kl} + \beta_{ij} e_{ij} \theta\right) \, dx \\
&\quad + \Lambda \int_{\Omega(y_3,x_3)} k_{ij} \theta_{,i} \theta_{,j} \, dx,
\end{aligned} \tag{5.4}$$

where the symmetries (2.5) and (2.24) are employed. Differentiation of (5.4) with respect to x_3 gives

$$H'(x_3) = \int_{D(x_3)} \left(c_{ijkl} e_{ij} e_{kl} + \beta_{ij} e_{ij} \theta\right) \, dS + \Lambda \int_{D(x_3)} k_{ij} \theta_{,i} \theta_{,j} \, dS, \tag{5.5}$$

in which we recall that a superposed prime indicates differentiation with respect to the argument. The Poincaré and the arithmetic-geometric mean inequalities lead to the lower bound

$$H'(x_3) \geq \left(1 - \frac{\gamma_1}{2}\right) \int_{D(x_3)} c_{ijkl} e_{ij} e_{kl} \, dS + \left(\Lambda - \frac{\widetilde{\beta}^2}{2\gamma_1 c_0 k_1 \lambda_1}\right) \int_{D(x_3)} k_{ij} \theta_{,i} \theta_{,j} \, dS, \tag{5.6}$$

where γ_1 is an arbitrary positive constant, k_1, $\widetilde{\beta}$, and c_0 are defined in (2.12), (2.18), and (2.26), respectively, and λ_1 denotes the first eigenvalue for the fixed membrane problem for the uniform cross-section D. On setting

$$\gamma_1^2 = \frac{\widetilde{\beta}^2}{\Lambda c_0 k_1 \lambda_1}, \tag{5.7}$$

and choosing Λ such that

$$0 < \gamma_1 < 2, \tag{5.8}$$

we conclude that the lower bound (5.6) becomes

$$H'(x_3) \geq \left(1 - \frac{\gamma_1}{2}\right) \left[\int_{D(x_3)} c_{ijkl} e_{ij} e_{kl} \, dS + \Lambda \int_{D(x_3)} k_{ij} \theta_{,i} \theta_{,j} \, dS \right]. \tag{5.9}$$

In consequence of (2.12) and (2.26), it follows that

$$H'(x_3) \geq 0, \qquad 0 \leq x_3 \leq \infty, \tag{5.10}$$

and by integration that

$$H(x_3) \geq \left(1 - \frac{\gamma_1}{2}\right) E(y_3, x_3) + H(y_3), \tag{5.11}$$

where the energy $E(y_3, x_3)$ is defined to be

$$E(y_3, x_3) = \int_{\Omega(y_3, x_3)} (c_{ijkl} e_{ij} e_{kl} + \Lambda k_{ij} \theta_{,i} \theta_{,j}) \, dx \geq 0, \qquad 0 \leq y_3 \leq x_3 \leq \infty. \tag{5.12}$$

Note that $H(x_3)$ is not necessarily either positive or negative. Moreover, subject to conditions (5.7) and (5.8), the relations (5.4) and (5.9) generate the upper bound

$$\begin{aligned} 0 &\leq \left(1 - \frac{\gamma_1}{2}\right) E(y_3, x_3), \qquad 0 \leq y_3 \leq x_3 \leq \infty, \\ &\leq H(x_3) - H(y_3) \\ &= \int_{\Omega(y_3, x_3)} (c_{ijkl} e_{ij} e_{kl} + \beta_{ij} e_{ij} \theta + \Lambda k_{ij} \theta_{,i} \theta_{,j}) \, dx, \end{aligned} \tag{5.13}$$

which is of later use.

The next task is to obtain an upper bound for the absolute value of $H(x_3)$ in terms of $H'(x_3)$. Since $n = (0, 0, 1)$ on $D(x_3)$, we have for all $x_3 \geq 0$,

$$\begin{aligned} |H(x_3)| &\leq \left| \int_{D(x_3)} t_{i3} u_i \, dS \right| + \Lambda \left| \int_{D(x_3)} q_3 \theta \, dS \right| \\ &= \left| \int_{D(x_3)} t_{ij} n_j u_i \, dS \right| + \Lambda \left| \int_{D(x_3)} q_i n_i \theta \, dS \right| \\ &= \left| \int_{D(x_3)} (c_{ijkl} e_{kl} n_j u_i + \beta_{ij} n_j u_i \theta) \, dS \right| \\ &\quad + \Lambda \left| \int_{D(x_3)} k_{ij} \theta_{,j} n_i \theta \, dS \right|, \end{aligned}$$

which, on using Schwarz's inequality, enables us to conclude that

$$
\begin{aligned}
|H(x_3)| \quad \leq \quad & \left[\int_{D(x_3)} c_{ijkl} e_{ij} e_{kl} \, dS \int_{D(x_3)} c_{ijkl} u_i u_k n_j n_l \, dS \right]^{1/2} \\
& + \widetilde{\beta} \left[\int_{D(x_3)} u_i u_i \, dS \int_{D(x_3)} \theta^2 \, dS \right]^{1/2} \\
& + \Lambda \left[\int_{D(x_3)} k_{ij} \theta_{,i} \theta_{,j} \, dS \int_{D(x_3)} k_{ij} n_i n_j \theta^2 \, dS \right]^{1/2} . \qquad (5.14)
\end{aligned}
$$

Furthermore, application of the arithmetic-geometric mean inequality and the Cauchy-Schwarz inequality leads to the bounds

$$
\begin{aligned}
\int_{D(x_3)} c_{ijkl} u_i u_k n_j n_l \, dS \quad &= \quad \int_{D(x_3)} c_{i3k3} u_i u_k \, dS \\
&\leq \quad \left[\int_{D(x_3)} c_{i3k3} u_i c_{p3k3} u_p \, dS \int_{D(x_3)} u_k u_k \, dS \right]^{1/2} \\
&\leq \quad \left[\int_{D(x_3)} (c_{i3k3} c_{i3k3}) u_i u_i \, dS \int_{D(x_3)} u_k u_k \, dS \right]^{1/2} \\
&\leq \quad \widetilde{c} \int_{D(x_3)} u_i u_i \, dS, \qquad (5.15)
\end{aligned}
$$

where

$$
\widetilde{c}^2 = \max_{\Omega} c_{i3k3} c_{i3k3}. \qquad (5.16)
$$

Similarly, we have

$$
\begin{aligned}
\int_{D(x_3)} k_{ij} n_i n_j \theta^2 \, dS \quad &= \quad \int_{D(x_3)} k_{33} \theta^2 \, dS \\
&\leq \quad k_3 \int_{D(x_3)} \theta^2 \, dS, \qquad (5.17)
\end{aligned}
$$

where

$$
k_3 = \max_{\Omega} k_{33}.
$$

Insertion of the bounds (5.15) and (5.17) into (5.14) succeeded by further application of the arithmetic-geometric mean inequality yields

$$
\begin{aligned}
|H((x_3)| \;\leq\; & \widetilde{c}\left[\int_{D(x_3)} c_{ijkl}e_{ij}e_{kl}\,dS \int_{D(x_3)} u_iu_i\,dS\right]^{1/2} \\
& +\widetilde{\beta}\left[\int_{D(x_3)} u_iu_i\,dS \int_{D(x_3)} \theta^2\,dS\right]^{1/2} \\
& +\Lambda\left[k_3\int_{D(x_3)} k_{ij}\theta_{,i}\theta_{,j}\,dS \int_{D(x_3)} \theta^2\,dS\right]^{1/2} \\
\leq\; & \left(\frac{\widetilde{c}^2\gamma_2}{2}\right)\int_{D(x_3)} c_{ijkl}e_{ij}e_{kl}\,dS \\
& +\frac{1}{2}\left(\frac{1}{\gamma_2}+\frac{1}{\gamma_3}\right)\int_{D(x_3)} u_iu_i\,dS \\
& +\frac{\widetilde{\beta}^2\gamma_3}{2}\int_{D(x_3)} \theta^2\,dS + \frac{\Lambda k_3^{1/2}}{(k_1\lambda_1)^{1/2}}\int_{D(x_3)} k_{ij}\theta_{,i}\theta_{,j}\,dS \\
\leq\; & \left\{\frac{\widetilde{c}^2\gamma_2}{2}\int_{D(x_3)} c_{ijkl}e_{ij}e_{kl}\,dS + \left(\frac{\widetilde{\beta}^2\gamma_3}{2k_1\lambda_1}+\frac{\Lambda k_3^{1/2}}{(k_1\lambda_1)^{1/2}}\right)\int_{D(x_3)} k_{ij}\theta_{,i}\theta_{,j}\,dS\right\} \\
& +\frac{1}{2}\left(\frac{1}{\gamma_2}+\frac{1}{\gamma_3}\right)\int_{D(x_3)} u_iu_i\,dS, \qquad (5.18)
\end{aligned}
$$

where Poincaré's inequality is used, the constants $\widetilde{\beta}, k_1, \lambda_1$ have previously been introduced, and γ_2, γ_3 are arbitrary positive constants chosen to satisfy

$$
\Lambda\left(\frac{\widetilde{c}^2\gamma_2}{2}\right) = \left(\frac{\widetilde{\beta}^2\gamma_3}{2k_1\lambda_1}+\frac{\Lambda k_3^{1/2}}{(k_1\lambda_1)^{1/2}}\right). \qquad (5.19)
$$

Consequently, by virtue additionally of (5.9), (5.18) reduces to

$$
|H(x_3)| \leq \frac{\widetilde{c}^2\gamma_2}{(2-\gamma_1)}H'(x_3) + \frac{1}{2}\left(\frac{1}{\gamma_2}+\frac{1}{\gamma_3}\right)\int_{D(x_3)} u_iu_i\,dS. \qquad (5.20)
$$

In order to bound the second term on the right of this inequality, we appeal to a generalised Korn's inequality stated in the following lemma, whose proof is variously established, for example, in [20], [12], or [3]. The proof, however, given in [12] holds only for sufficiently large x_3.

Lemma 5.1 *Let Ω be the semi-infinite cylinder defined in this Section, and let $v_i(x)$ be a smooth vector function defined on Ω that satisfies the lateral boundary conditions*

$$
v_i(x) = 0, \qquad x \in \partial D \times [0,\infty).
$$

Then there exists a positive bounded computable positive constant N dependent on the geometry of the (prismatic) cross-section D, such that for $0 \leq y_3 < x_3 \leq$

∞*, the following inequality holds*

$$\int_{D(y_3)} v_i v_i \, dS + \int_{D(x_3)} v_i v_i \, dS \leq N \int_{\Omega(y_3,x_3)} (v_{i,j} + v_{j,i}) (v_{i,j} + v_{j,i}) \, dx. \quad (5.21)$$

Substitution of (5.21) in (5.20) subsequently leads to the successive inequalities

$$\begin{aligned}
|H(x_3)| &\leq \frac{\widetilde{c}^2 \gamma_2}{(2-\gamma_1)} H'(x_3) + \frac{1}{2}\left(\frac{1}{\gamma_2} + \frac{1}{\gamma_3}\right) \left[4N \int_{\Omega(0,x_3)} e_{ij} e_{ij} \, dx \right] \\
&\leq \frac{\widetilde{c}^2 \gamma_2}{(2-\gamma_1)} H'(x_3) + \frac{1}{2}\left(\frac{1}{\gamma_2} + \frac{1}{\gamma_3}\right) \left[\frac{4N}{c_0} \int_{\Omega(0,x_3)} c_{ijkl} e_{ij} e_{kl} \, dx \right] \\
&\leq \frac{\widetilde{c}^2 \gamma_2}{(2-\gamma_1)} H'(x_3) + \frac{1}{2}\left(\frac{1}{\gamma_2} + \frac{1}{\gamma_3}\right) \frac{4N}{c_0} E(0,x_3) \\
&\leq \frac{\widetilde{c}^2 \gamma_2}{(2-\gamma_1)} H'(x_3) \\
&\quad + \left(\frac{1}{\gamma_2} + \frac{1}{\gamma_3}\right) \frac{4N}{c_0(2-\gamma_1)} \left(H(x_3) - H(0)\right), \qquad (5.22)
\end{aligned}$$

where (2.12), (2.26), and (5.11) are used. The basic differential inequality (5.22) is now employed to establish alternative evolutionary growth and decay properties descriptive of spatial stability. The mutually exclusive initial data for which $H(0) > 0$ or $H(0) \leq 0$ require separate discussion.

5.3 Growth and decay estimates

We consider first the case for which $H(0) > 0$ and prove the following the theorem:

Theorem 5.1 *Suppose initial data is such that* $H(0) > 0$*. Then*

$$H(x_3) \geq H(0) \exp(\gamma x_3), \qquad 0 \leq x_3 \leq \infty, \quad (5.23)$$

where

$$\gamma = \frac{(2-\gamma_1)}{2\widetilde{c}^2 (e\gamma_3 + f)}, \quad (5.24)$$

and γ_1 *and* $\widetilde{c}$ *are defined in (5.7) and (5.16), while* γ_3, e, f *are given by expressions (5.28) and (5.27) below.*

Proof: From (5.10), the derivative of $H(x_3)$ with respect to x_3 is non-negative, and consequently assumption $H(0) > 0$ implies that $H(x_3) \geq H(0) > 0$ for $x_3 \geq 0$, so that inequality (5.22) may be written as

$$H(x_3) \leq \frac{\widetilde{c}^2 \gamma_2}{(2-\gamma_1)} H'(x_3) + \left(\frac{1}{\gamma_2} + \frac{1}{\gamma_3}\right) \frac{4N}{c_0(2-\gamma_1)} H(x_3), \quad (5.25)$$

for $0 \leq x_3 \leq \infty$.

Recall that the arbitrary positive constants γ_2, γ_3 have been chosen to satisfy condition (5.19), which we rewrite in the form

$$\gamma_2 = e\gamma_3 + f, \tag{5.26}$$

where

$$e = \frac{\widetilde{\beta}^2}{\Lambda \widetilde{c}^2 k_1 \lambda_1}, \qquad f = \frac{2k_3^{1/2}}{\widetilde{c}^2 (k_1 \lambda_1)^{1/2}}. \tag{5.27}$$

Observe also that the positive constant γ_1 and the constant Λ are determined by conditions (5.7) and (5.8). We complete the choice of γ_2 and γ_3 by imposing the condition

$$2g(\gamma_2 + \gamma_3) = \gamma_2 \gamma_3,$$

where

$$g = \frac{4N}{c_0(2 - \gamma_1)}.$$

Accordingly, γ_3 is given by

$$\gamma_3 = \frac{[2g(1+e) - f] + \sqrt{\left([2g(1+e) - f]^2 + 8efg\right)}}{2e}, \tag{5.28}$$

and γ_2 follows from (5.26). We recover (5.23) after substitution in (5.25) followed by integration. □

An appeal to identity (5.5) establishes the following corollary to Theorem 5.1, which provides an alternative interpretation of the growth condition.

Corollary 5.1 *The total energy (5.12) is unbounded as* $x_3 \to \infty$ *when* $H(0) > 0$.

Proof: The arithmetic-geometric mean inequality applied to the second term on the right of (5.5) leads to the inequality, compatible with the bound (5.11),

$$H(x_3) \leq \left[1 + \frac{1}{2}\left(\frac{\widetilde{\beta}^2}{c_0 \Lambda \lambda_1 k_1}\right)^{1/2}\right] E(y_3, x_3) + H(y_3), \qquad 0 \leq y_3 < x_3 \leq \infty, \tag{5.29}$$

which implies the conclusion since we may let $y_3 \to 0$ and $x_3 \to \infty$. □

We next consider the class of displacements and temperatures for which the total energy $E(0, \infty)$ is bounded and immediately have:

Proposition 5.1 *Within the class of non-trivial displacements and temperatures, a bounded total energy implies* $H(x_3) < 0$ *for* $x_3 \geq 0$.

Proof: Let us first consider the case $H(0) = 0$, and suppose that $H(x_3) \equiv 0$ for all $x_3 \geq 0$. Then it follows from inequality (5.11) that $e_{ij} = \theta_{,i} = 0$ for $x_3 \geq 0$. Consequently, by virtue of the lateral boundary conditions, the corresponding displacement and temperature are identically zero and therefore excluded by assumption. Next suppose that for $H(0) = 0$ we have $H(z) > 0$ for some $x_3 = z > 0$. Then by Theorem 5.1, the energy is unbounded for $x_3 \geq z$, contrary to hypothesis. Consequently, we conclude that $H(x_3) < 0$ for $x_3 \geq 0$ whenever the total energy is bounded. □

The conditions for Proposition 5.1 are now employed to derive a decay estimate.

Theorem 5.2 *In the class of displacements and temperatures that possess bounded total energy, the energy flux function $H(x_3)$ satisfies the differential inequality*

$$0 \leq H'(x_3) + \gamma H(x_3), \qquad x_3 \geq 0, \tag{5.30}$$

where γ is specified by (5.24). Upon integration, we therefore have

$$-H(x_3) \leq -H(0) \exp(-\gamma x_3), \qquad x_3 \geq 0. \tag{5.31}$$

Proof: We repeat the derivation of inequality (5.22) apart from one, but important, difference.

A total bounded energy implies that $H(x_3) < 0, x_3 \geq 0$, and upon returning to inequality (5.20), we have

$$-H(x_3) \leq \frac{\widetilde{c}^2 \gamma_2}{(2-\gamma_1)} H'(x_3) + \frac{1}{2}\left(\frac{1}{\gamma_2} + \frac{1}{\gamma_3}\right) \int_{D(x_3)} u_i u_i \, dS. \tag{5.32}$$

where the constants γ_1, Λ are assumed to satisfy (5.7) and (5.8), and γ_2, γ_3 are given by (5.28) and (5.26). Now successively apply the bounds (5.21) and (5.11) not to the region $\Omega(y_3, x_3)$ but to the region $\Omega(x_3, z_3)$ where $0 \leq x_3 < z_3$.We obtain:

$$\begin{aligned}
\int_{D(x_3)} u_i u_i \, dS &\leq \frac{4N}{c_0} \int_{\Omega(x_3,z_3)} c_{ijkl} e_{ij} e_{kl} \, dx \\
&\leq \frac{4N}{c_0} E(x_3, z_3) \\
&\leq \frac{8N}{c_0(2-\gamma_1)} \left[H(z_3) - H(x_3)\right] \\
&\leq \frac{8N}{c_0(2-\gamma_1)} \left(-H(x_3)\right),
\end{aligned} \tag{5.33}$$

as $H(z_3) < 0$. On insertion of (5.33) into (5.32) and after letting γ_3 and γ_2 be determined respectively from (5.28) and (5.26), we conclude that (5.30) holds. Integration leads to (5.31) □.

The conclusion of Theorem 5.2 may be expressed in terms of the energy $E(x_3, \infty)$. We have:

Proposition 5.2 *The energy $E(x_3, \infty)$, defined by (5.12), possesses the exponentially decaying upper bound*

$$E(x_3, \infty) \leq M E(0, \infty) \exp(-\gamma x_3), \qquad x_3 \geq 0, \tag{5.34}$$

where the positive constant M is given by

$$M = \left(1 - \frac{\gamma_1}{2}\right)^{-1} \left[1 + \frac{1}{2}\left(\frac{\widetilde{\beta}^2}{c_0 \Lambda \lambda_1 k_1}\right)^{1/2}\right].$$

Proof: Let $x_3 \to \infty$ in the upper bound (5.31) to conclude that

$$H(x_3) \to 0 \qquad \text{as } x_3 \to \infty,$$

which upon insertion into inequality (5.11) leads to

$$-H(x_3) \geq \left(1 - \frac{\gamma_1}{2}\right) E(x_3, \infty), \qquad x_3 \geq 0.$$

Moreover, inequality (5.29), which remains valid under the present conditions, implies

$$-H(0) \leq \left[1 + \frac{1}{2}\left(\frac{\widetilde{\beta}^2}{c_0 \Lambda \lambda_1 k_1}\right)^{1/2}\right] E(0, \infty)$$

and in consequence, the bound (5.34) is recovered. □

5.4 Uniqueness

Uniqueness of the displacement and temperature is immediate from the last Proposition.

Corollary 5.2 *For the Dirichlet boundary value problem defined on the semi-infinite prismatic cylinder Ω, within the class of displacements and temperature for which the total energy is bounded, there is at most only one solution.*

Remark 5.1 *Other boundary conditions may be similarly treated, although for Neumann boundary conditions, uniqueness holds only to within rigid body displacements and constant temperatures.*

Proof: Without loss, the end base displacement and temperature may be assumed to vanish so that from the definition, we have that $H(0) = 0$, and consequently (5.31) implies that $H(x_3) = 0, x_3 \geq 0$. We then have from (5.11) that $E(0, \infty) = 0$ and the conclusion follows by standard arguments. □

5.5 Decay estimate in terms of base data

Decay estimates for the solution measured either by the cross-section energy flux or by energy stored in a partial volume are established in Section 5.3. However, the practical utility of these estimates is complete only when the amplitude functions, either $H(0)$ or $E(0, \infty)$, are bounded in terms of base data. We derive such a bound for $E(0, \infty)$ subject to Dirichlet base data, and for this purpose recall the continuously differentiable functions $v_1(x), \phi(x)$ introduced in Section 4.4 and defined to possess the properties (4.22)-(4.24). Let

$$\begin{aligned} e_{ij}(v) &= \frac{1}{2}\left(v_{i,j} + v_{j,i}\right), \\ E^{(v)}(y_3, x_3) &= \int_{\Omega(y_3,x_3)} c_{ijkl} e_{ij}(v) e_{kl}(v)\, dx \\ &\quad + \Lambda \int_{\Omega(y_3,x_3)} k_{ij} \phi_{,i} \phi_{,j}\, dx, \end{aligned}$$

where Λ satisfies the condition inherent in (5.8).

On supposing that all source terms vanish, we multiply (2.25) by v_i, add to (2.11) multiplied by $\Lambda\phi$, integrate by parts over Ω, and note the upper bound (5.13) to obtain

$$\begin{aligned} \left(1 - \frac{\gamma_1}{2}\right) E(0,\infty) &\leq \int_\Omega \left(c_{ijkl} e_{ij}(v) e_{kl} + \beta_{ij} e_{ij}(v)\theta + \Lambda k_{ij} \theta_{,i} \phi_{,j}\right) dx \\ &\leq \frac{\gamma_4}{2} \int_\Omega c_{ijkl} e_{ij} e_{kl}\, dx \\ &\quad + \left[\left(\frac{\widetilde{\beta}^2}{c_0 \lambda_1 k_1}\right)^{1/2} \frac{\gamma_5}{2} + \frac{\Lambda\gamma_6}{2}\right] \int_\Omega k_{ij} \theta_{,i} \theta_{,j}\, dx \\ &\quad + \left[\frac{1}{2\gamma_5}\left(\frac{\widetilde{\beta}^2}{c_0 \lambda_1 k_1}\right)^{1/2} + \frac{1}{2\gamma_4}\right] \int_\Omega c_{ijkl} e_{ij}(v) e_{kl}(v)\, dx \\ &\quad + \frac{\Lambda}{2\gamma_6} \int_\Omega k_{ij} \phi_{,i} \phi_{,j}\, dx, \end{aligned} \tag{5.35}$$

where the Schwarz, Poincaré, and arithmetic-mean inequalities are employed, and $\gamma_4, \gamma_5, \gamma_6$ are arbitrary positive constants chosen to satisfy

$$\gamma_4 = \left(1 - \frac{\gamma_1}{2}\right), \tag{5.36}$$

$$\begin{aligned} \gamma_5 &= \gamma_6 \\ &= \Lambda\left(1 - \frac{\gamma_1}{2}\right)\left[\Lambda + \left(\frac{\widetilde{\beta}^2}{c_0 \lambda_1 k_1}\right)^{1/2}\right]^{-1}. \end{aligned} \tag{5.37}$$

In consequence, inequality (5.35) yields

$$\left(1 - \frac{\gamma_1}{2}\right) E(0,\infty) \leq c_{11} \int_\Omega c_{ijkl} e_{ij}(v) e_{kl}(v)\, dx + \Lambda c_{12} \int_\Omega k_{ij} \phi_{,i} \phi_{,j}\, dx,$$

where the constants c_{11}, c_{12} are given by

$$\begin{aligned} c_{11} &= \left[\frac{1}{\gamma_4} + \frac{1}{\gamma_5}\left(\frac{\widetilde{\beta}^2}{c_0 \lambda_1 k_1}\right)^{1/2}\right], \\ c_{12} &= \frac{1}{\gamma_5}, \end{aligned}$$

and γ_4, γ_5 are given by (5.36) and (5.37).

Upon selecting the functions v_i and ϕ to be given by (4.26) and (4.27), the upper bound becomes

$$\left(1 - \frac{\gamma_1}{2}\right) E(0, \infty) \leq \frac{1}{2s} D_9 - D_{10} + \frac{s}{2} D_{11},$$

in which

$$\begin{aligned} D_9 &= (c_{11} D_3 + \Lambda c_{12} D_6), \\ D_{10} &= (c_{11} D_4 + \Lambda c_{12} D_7), \\ D_{11} &= (c_{11} D_5 + \Lambda c_{12} D_8), \end{aligned}$$

and

$$\begin{aligned} D_3 &= \int_{D(0)} c_{i\alpha j\beta} w_{i,\alpha} w_{j,\beta} \, dS, \\ D_4 &= \int_{D(0)} c_{i\alpha j3} w_{i,\alpha} w_j \, dS, \\ D_5 &= \int_{D(0)} c_{i3j3} w_i w_j \, dS, \\ D_6 &= \int_{D(0)} k_{\alpha\beta} \chi_{,\alpha} \chi_{,\beta} \, dS, \\ D_7 &= \int_{D(0)} k_{\alpha 3} \chi_{,\alpha} \chi \, dS, \\ D_8 &= \int_{D(0)} k_{33} \chi^2 \, dS. \end{aligned}$$

Optimisation with respect to s produces a final upper bound represented by

$$\left(1 - \frac{\gamma_1}{2}\right) E(0, \infty) \leq \left[\sqrt{(D_9 D_{11})} - D_{10}\right], \tag{5.38}$$

which implies and improves the bound (4.28) determined by similar, but less exact, arguments in Section 4.4. Standard inequalities demonstrate that the expression on the right of (5.38) is non-negative. Indeed, because $w_i(x_\alpha), \chi(x_\alpha)$

are independent of x_3, we have

$$
\begin{aligned}
D_4 &= \int_{D(0)} c_{ijkl} w_{i,j} w_k n_l \, dS \\
&= \frac{1}{4} \int_{D(0)} c_{ijkl} \left(w_{i,j} + w_{j,i}\right) \left(w_k n_l + w_l n_k\right) \, dS \\
&\leq \frac{1}{4} \left[\int_{D(0)} c_{ijkl} \left(w_{i,j} + w_{j,i}\right) \left(w_{k,l} + w_{l,k}\right) \, dS \right]^{1/2} \\
&\quad \times \left[\int_{D(0)} c_{ijkl} \left(w_i n_j + w_j n_i\right) \left(w_k n_l + w_l n_k\right) \, dS \right]^{1/2} \\
&= \left[\int_{D(0)} c_{ijkl} w_{i,j} w_{k,l} \, dS \int_{D(0)} c_{ijkl} w_i n_j w_k n_l \, dS \right]^{1/2} \\
&= \left[\int_{D(0)} c_{i\alpha k\beta} w_{i,\alpha} w_{k,\beta} \, dS \int_{D(0)} c_{i3k3} w_i w_k \, dS \right]^{1/2} \\
&= D_3^{1/2} D_5^{1/2}.
\end{aligned}
$$

Similarly, we have

$$
D_7 \leq D_6^{1/2} D_8^{1/2}.
$$

But by the arithmetic-geometric mean inequality, it follows that

$$
2 \left(D_3 D_5 D_6 D_8\right)^{1/2} \leq \left(D_5 D_6 + D_3 D_8\right),
$$

and consequently

$$
\left(c_{11} D_4 + \Lambda c_{12} D_7\right)^2 \leq \left(c_{11} D_3 + \Lambda c_{12} D_6\right) \left(c_{11} D_5 + \Lambda c_{12} D_8\right),
$$

which implies that the right side of (5.38) is non-negative.

6 Further comment

The method outlined in the previous section may be applied to unbounded bodies contained between parallel planes for which Poincaré's inequality remains valid in the form used. For other types of unbounded bodies, Poincaré's inequality is replaced by Wirtinger's inequality in the construction of a differential inequality with respect to measures taken over spherical or other suitable curvilinear surfaces. An application to isothermal elasticity presented in [13] demonstrates that decay is characteristically algebraic and not exponential, but whether such behaviour generally occurs in corresponding thermoelastic problems awaits clarification. Exact solutions, however, derived in [15] to certain problems in isotropic thermoelasticity confirm that for these particular problems the decay is algebraic.

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